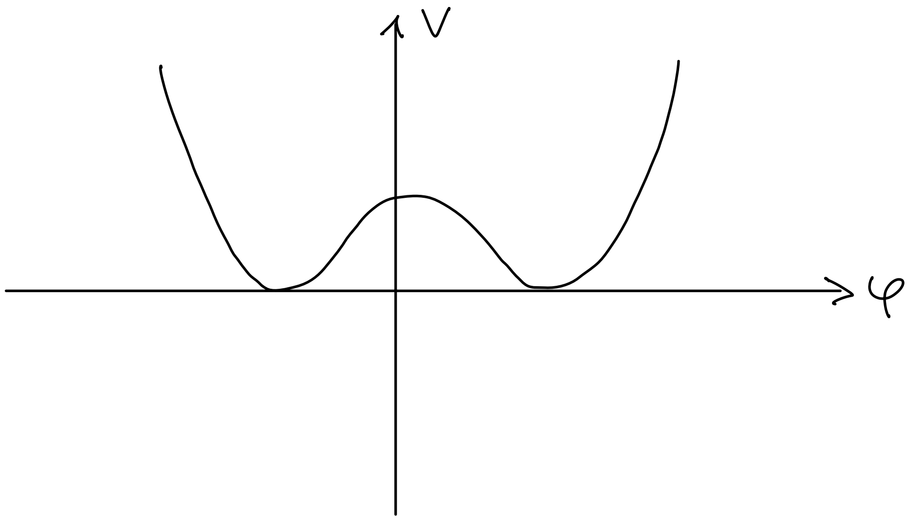
§5.3 Solitons

Consider the toy model $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$ with double-well potential

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - v^2)^2 \qquad (1)$$

in (1+1)-dim. spacetime



→ two vacua $\varphi = \pm v$

Pick one and study oscillations around it: $\varphi = v + \chi$ "symmetry breaking" (will study in QFT 2)

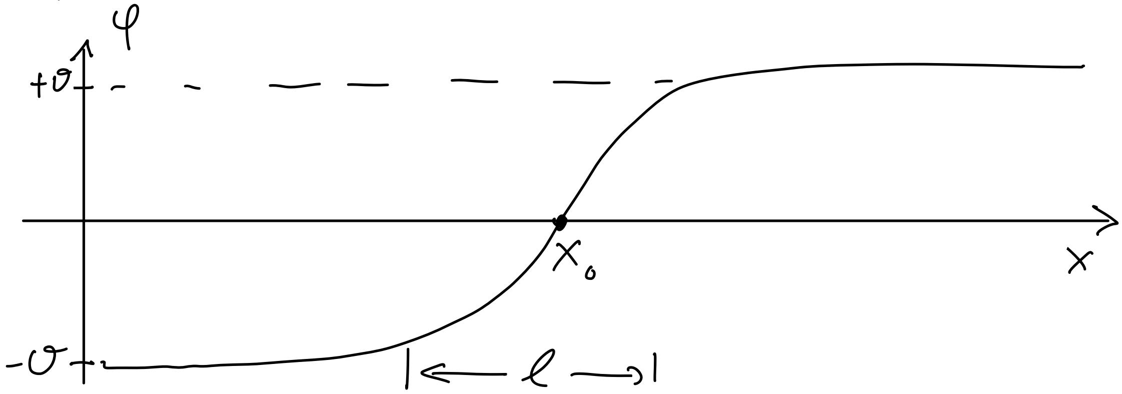
→ expanding $\mathcal{L}$ in χ, one finds χ describes particle with mass $m = (\lambda v^2)^{\frac{1}{2}}$

From now on denote x as space and t as time

We can have a <u>time independent</u> field configuration with

$$\varphi(x) \longrightarrow -v \quad \text{as} \quad x \longrightarrow -\infty$$
$$\varphi(x) \longrightarrow +v \quad \text{as} \quad x \longrightarrow +\infty$$

and changing from $-v$ to $+v$ around some point x_0:

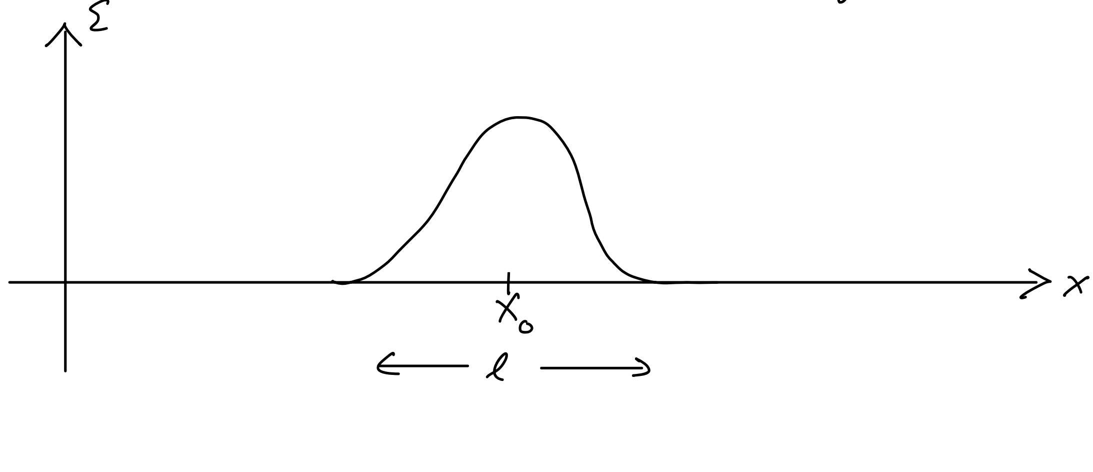


"domain wall"

$\rightarrow$ energy per unit length is

$$\mathcal{E}(x) = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2 \quad (2)$$

For the domain wall this gives:



$\rightarrow$ the total energy or mass is then:

$$M = \int dx\, \mathcal{E}(x)$$

consisting of the two contributions:

- "spatial variation":

$$\int dx \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 \sim \ell\left(\frac{v}{\ell}\right)^2 \sim v^2/\ell$$

- "potential energy":

$$\int dx\, \lambda(\varphi^2 - v^2)^2 \sim \ell \lambda v^4$$

Minimizing the energy gives:

$$\frac{dM}{d\ell} = 0 \sim -\frac{v^2}{\ell^2} + \lambda v^4$$

$$\Rightarrow \frac{v^2}{\ell} \sim \ell \lambda v^4$$

$$\rightarrow \ell \sim (\lambda v^2)^{-\frac{1}{2}} \sim \frac{1}{\mu} \longrightarrow \text{mass: } M \sim \mu \frac{\mu^2}{\lambda}$$

Thus we get a "lump of energy" spread over a region of length $\ell \sim \frac{1}{\mu}$

$\longrightarrow$ can boost to any velocity by Lorentz inv.

"soliton" or "kink"

Topological stability

The kink is "topologically stable" as it would cost an infinite amount of energy to lift $\varphi(x)$ over the potential barrier from x_0 to $+\infty$

$\longrightarrow$ in two dimensions have conserved current:

$$J^\mu = \frac{1}{2v} \varepsilon^{\mu\nu} \partial_\nu \varphi \quad \text{"topological current"}$$

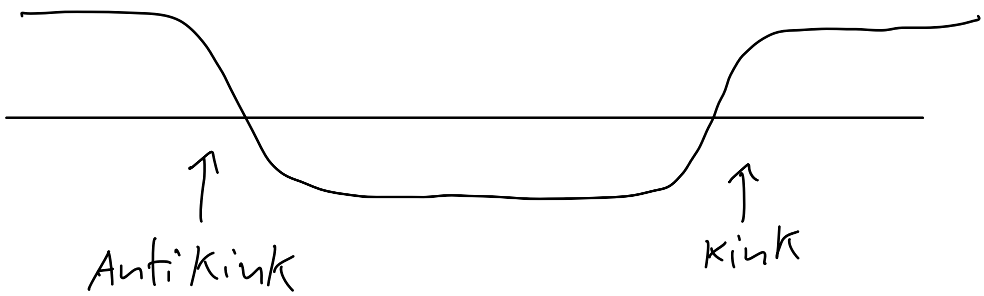
with charge

$$Q = \int_{-\infty}^{\infty} dx \, J^0(x) = \frac{1}{2v} \left[\varphi(+\infty) - \varphi(-\infty) \right] = 1$$

ordinary scaler particle has charge $Q=0$ $\longrightarrow$ Kink cannot decay to ordinary scala particles!

antikinks: $Q = -1 \quad \varphi(-\infty) = +v, \; \varphi(+\infty) = -v$

$\longrightarrow$ Kink and antikink can annihilate into scalar particles



Non-perturbative phenomena

Solitons (kinks) are examples of "non-perturbative" phenomena which are outside the range of perturbation theory

($M \sim \mu \frac{\mu^2}{\lambda}$ $\longrightarrow$ not detectable for $\lambda \ll 1$)

more precisely,

$$M = \int dx \left[\underbrace{\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2}_{=:a} + \underbrace{\frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2}_{=:b} \right]$$

$\varphi(x) \to v f(y)$

$y = \mu x$

$$= \left(\frac{\mu^2}{\lambda}\right) \mu \int dy \left[\frac{1}{2}\left(\frac{df}{dy}\right)^2 + \frac{1}{4}\left(f^2 - 1\right)^2 \right]$$

pure number

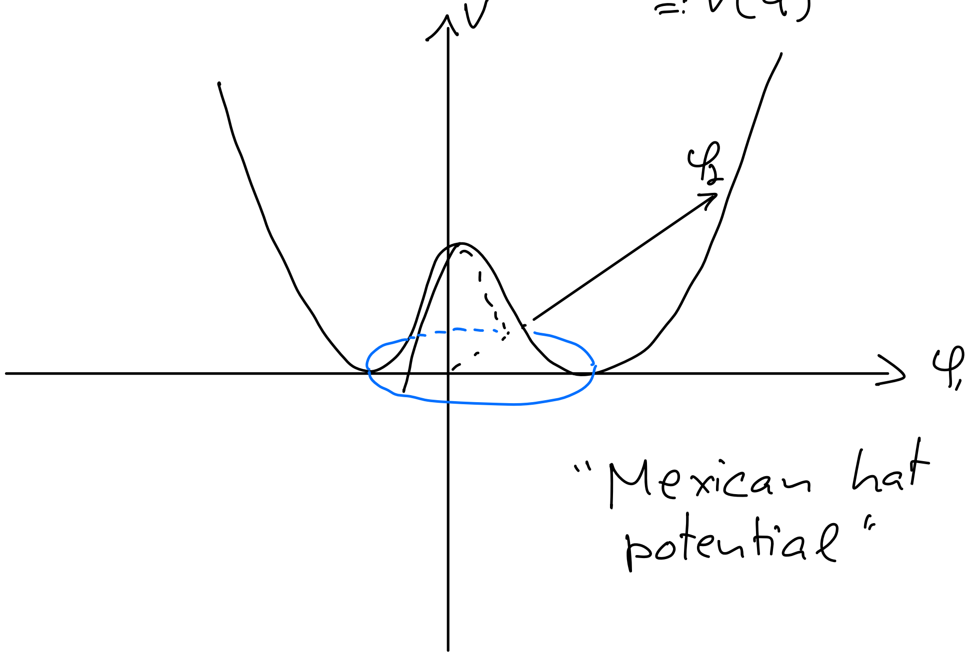
$$= a \left(\frac{\mu^2}{\lambda}\right) \mu$$

Vortices

Vortices are yet another example of solitonic field configurations

Consider complex scalar field in (2+1)-dim. spacetime with

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - \underbrace{\lambda(\varphi^\dagger\varphi - v^2)^2}_{=: V(\varphi)}$$



"Mexican hat potential"

We look for time independent finite energy configurations with

$$M = \int d^2x \left[\partial_i \varphi^\dagger \partial_i \varphi + \lambda(\varphi^\dagger\varphi - v^2)^2\right] < \infty$$

$\longrightarrow$ $|\varphi| \longrightarrow v$ at $|\vec{x}| \longrightarrow \infty$

Consider the Ansatz $\varphi \longrightarrow v e^{i\theta}$ in polar coordinates

$\longrightarrow$ $\varphi = \varphi_1 + i\varphi_2$ gives $(\varphi_1, \varphi_2) = v(\cos\theta, \sin\theta)$

Recall the $SO(2)$-current (§3.1)

$$J_i = i(\partial_i \varphi^\dagger \varphi - \varphi^\dagger \partial_i \varphi)$$

$$= i\left[\partial_i(\varphi_1 - i\varphi_2)(\varphi_1 + i\varphi_2) - (\varphi_1 - i\varphi_2)\partial_i(\varphi_1 + i\varphi_2)\right]$$

$$= 2\left[-\varphi_2 \partial_i \varphi_1 + \varphi_1 \partial_i \varphi_2\right]$$

$\longrightarrow$ current whirls about at spatial infinity

"vortex", topological as $\varphi \xrightarrow[r\to\infty]{} v e^{im\theta} \in \pi_1(S^1)$ for $m \in \mathbb{Z}$

$\longrightarrow$ have $\partial_i \varphi \sim v\left(\frac{1}{r}\right)$ as $r \to \infty$

$\longrightarrow$ for the term $\partial_i \varphi^\dagger \partial_i \varphi$ we then get

$$\int d^2x\, \partial_i \varphi^\dagger \partial_i \varphi \xrightarrow{r\to\infty} v^2 \int d^2x \frac{1}{r^2}$$

energy diverges logarithmically

cure:

Gauge the theory: $\partial_i \varphi \longrightarrow D_i \varphi = \partial_i \varphi - ieA_i \varphi$

Now require $A_i \xrightarrow{r\to\infty} \underbrace{-\frac{i}{e}\frac{1}{|\varphi|^2}\varphi^\dagger \partial_i \varphi}_{=\frac{1}{e}\partial_i \theta}$ so that $D_i \varphi \overset{r\to\infty}{\sim} 0$

$\longrightarrow$ flux $= \int d^2x\, F_{12} = \oint dx_i A_i = \frac{2\pi}{e}$ vortex carries mag. flux "flux tube"

Monopoles

Let us now move to (3+1)-dimensions

→ spacial infinity is S^2

Consider Lagrangian with $O(3)$ symmetry:

$$\mathcal{L} = \frac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} - V(\vec{\varphi}\cdot\vec{\varphi}) \;, \quad \vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$$

Let us choose $V = \lambda(\vec{\varphi}^2 - v^2)^2$

$$M = \int d^3x \left[\frac{1}{2}(\partial\vec{\varphi})^2 + \lambda(\vec{\varphi}^2 - v^2)^2\right]$$

→ finiteness forces $|\vec{\varphi}| = v$ at $r \to \infty$

→ obtain a map $S^2 \to S^2$

$$\frac{x^a}{r} \underset{r\to\infty}{\longrightarrow} v\,\frac{x^a}{r}$$

finiteness requirement for first term forces us to introduce $O(3)$ gauge potential A^b_μ:

$$\partial_i\varphi^a \longmapsto D_i\varphi^a = \partial_i\varphi^a + e\varepsilon^{abc}A_i^b\varphi^c$$

→ choose A^b_μ such that $D_i\varphi^a \underset{r\to\infty}{\sim} 0$:

$$A^b_i \xrightarrow[r\to\infty]{} \frac{1}{e}\,\varepsilon^{bij}\,\frac{x^j}{r^2}$$

solution is time-independent $\rightarrow A_0^b = 0$

$A_i^b \rightarrow$ magnetic $\vec{B}$ field pointing in radial direction!

"'t Hooft-Polyakov monopole" solution

flux $\int d\vec{S}\cdot\vec{B}$ is again quantized as for the Dirac monopole

mass is given by

$$M = \int d^3x \left[\frac{1}{4}(F_{ij})^2 + \frac{1}{2}(D_i \vec{\varphi})^2 + V(\vec{\varphi}) \right]$$

we have

$$\frac{1}{4}(\vec{F}_{ij})^2 + \frac{1}{2}(D_i\vec{\varphi})^2 = \frac{1}{4}\left(\vec{F}_{ij} \pm \varepsilon_{ijk} D_k \vec{\varphi}\right)^2$$
$$\mp \frac{1}{2}\varepsilon_{ijk}\vec{F}_{ij}\cdot D_k\vec{\varphi}$$

$$\rightarrow \quad M \geq \int d^3x \left[\mp \frac{1}{2}\varepsilon_{ijk}\vec{F}_{ij}\cdot D_k\vec{\varphi} + V(\vec{\varphi}) \right]$$

and

$$\int d^3x \, \frac{1}{2}\varepsilon_{ijk}\vec{F}_{ij}\cdot D_k\vec{\varphi} = \int d^3x \, \frac{1}{2}\varepsilon_{ijk}\,\partial_k\left(\vec{F}_{ij}\cdot\vec{\varphi}\right)$$
$$= v\int d\vec{S}\cdot\vec{B} = 4\pi v g$$

for $|\vec{\varphi}| \xrightarrow[r\to\infty]{} v$, $V(\vec{\varphi}) \sim 0$

$\rightarrow \quad M = 4\pi v g$ for $\vec{F}_{ij} = \pm\varepsilon_{ijk} D_k\vec{\varphi}$

"Bogomol'nyi-Prasad-Sommerfeld" or "BPS"-state